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The Public Good Index with Threats in A Priori Unions

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Abstract We consider four variants of the Public Good Index for games with a priori unions. The first variant extends the original idea of Holler (1982) assuming that the coalitional value is a public good and only minimal winning coalitions are relevant. The remaining three variants assign power in a two-step way. In the first step, power is distributed between unions according to the PGI of the quotient game. On the member stage, the indices take into account the possibilities of players to threaten their partners through leaving their union. We discuss the four indices and compare them to two extensions previously introduced in Alonso-Meijide et al. (2008). Theoretical reasoning and numerical examples demonstrate the various measures may differ substantially.

Keywords simple game, coalition structure, Public Good Index

1. Introduction

Owen (1977) proposed an extension of the Shapley value to games with a priori unions, i.e., a partition of the set of players which describes an a priori coalition structure. In the first step, this measure distributes the total value of a winning coalition to the a priori unions in accordance with the Shapley

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value. In the second step, once again applying the Shapley value, the total reward of a union is allocated among its members, taking into account the possibility of their joining other unions. Using similar reasoning, Owen (1982) proposed a version of the Banzhaf value for measuring power in TU games with a priori unions. More recently, Alonso-Meijide et al. (2009) extended the Deegan-Packel index to games with a priori unions. In a second paper, Alonso-Meijide et al. (2008) discuss two characterizations of the Public Good Index for games with a priori unions. The first alternative, the Solidarity PGI, stresses the public good property which suggests that all members of a winning coalition derive equal power. The second one follows Owen (1977, 1982) and refers to essential coalitions when allocating power shares.

The following paper considers further variants of applying the PGI to games with a priori unions which differ with respect to the threat potential a player has by his or her possibility to join players or sets of players outside the a priori union that defines the membership in the game. The first variant builds on Holler (1982) assuming that the coalition is a public good and only minimal winning coalitions are relevant when it comes to measuring power. Here threats have no impact on the distribution of the coalition value as the coalition value is a pure public good and exclusion and rivalry in consumption do not apply. The remaining three variants assign power in a two-step way. In the first step, power is distributed between unions according to the PGI of the quotient game. On the member stage, the indices take into account the possibilities of players to threaten their partners through leaving their union.

Obviously, the discussed variants constitute different solution concepts for coalition games with a priori unions. The discussion will demonstrate that 'different solution concepts can...be thought of as results of choosing not only which properties one likes, but also which examples one wishes to avoid' Aumann (1977, p.471).

The paper is organized as follows. In Section 2, we provide the analytical tools such as simple games, power indices, and coalition structures. Section 3 introduces the four new coalitional Public Good Indices. In Section 4, we investigate basic properties and the relationships of the overall six coalitional PGIs.

2. Preliminaries

2.1 *Simple Games and the Public Good Index*

A *simple game* is a pair (N,W) of the finite *set of players* $N = \{1,\ldots,n\}$ and the *set of winning coalitions* W satisfying $\emptyset \notin W, N \in W$ and $S \in W \Rightarrow T \in W$ for all $S \subseteq T$. A coalition $S \subseteq N$ is called *winning* or *losing* according to whether $S \in W$ or $S \notin W$. A winning coalition S is a *minimal winning coalition* (MWC) if each proper subset $T \subset S$ is a losing coalition. We denote the *set of minimal winning coalitions* by M. Since M contains all relevant information and is more suitable for what follows, we mainly denote the game (N,W) by the equivalent description (N,M) throughout this work.

A *power index* is a mapping f assigning each simple game (N,M) an n-dimensional real valued vector $f(N,M) = (f_1(N,M),\ldots,f_n(N,M))$. Based on the assumptions that coalitional values are public goods and only minimal winning coalitions are relevant when it comes to power, the *Public Good Index* (PGI) proposed by Holler (1982) assigns power proportional to the number of minimal winning coalitions a player belongs to. Denoting M_i as the set of minimal winning coalitions containing i, the PGI δ is given by

$$\delta_i(N,M) = \frac{|M_i|}{\sum_j |M_j|}, \quad i = 1,\ldots,n. \tag{2.1}$$

Holler and Packel (1983) characterize the PGI as the unique power index satisfying efficiency, symmetry, null player, and PGI-mergeability, the axioms being defined as follows. An index f satisfies *efficiency* if $\sum_i f_i(N,M) = 1$ for all simple games (N,M). Two players i and j are *symmetric* if $S \cup \{i\} \in W \Leftrightarrow S \cup \{j\} \in W$ for all $S \not\ni i, j$. A player i is called a *null player* if $S\backslash\{i\} \in W$ for all coalitions $S \in W$. Power index f satisfies *symmetry* if $f_i(N,M) = f_j(N,M)$ for all symmetric players i and j and *null player* if it holds $f_i(N,M) = 0$ for all null players i. Two simple games (N,M) and (N,M') are *mergeable* if $S \in M$ implies $S \notin W'$ and $S \in M'$ implies $S \notin W$. In particular, the sets of minimal winning coalitions M and M' are disjoint. The *merged game* $(N, M \oplus M')$ of two mergeable games (N,M) and (N,M') is defined by $M \oplus M' = M + M'$. Now, a power index f satisfies *PGI-mergeability* if for all mergeable games (N,M) and (N,M') it holds $\sum_i |M_i + M'_i| f(N, M \oplus M') = \sum_i |M_i| f(N,M) + \sum_i |M'_i| f(N,M')$.

2.2 *Simple games with coalition structures*

For a set of players N, a *coalition structure* is a partition $P = \{P_1,\ldots,P_p\}$ of N, that is, a set of nonempty and mutually disjoint subsets of N whose union coincides with N. We also use P as the mapping assigning each player i the *union* $P(i) \in P$ he is a member of. A *simple game with coalition structure* is a triplet (N,W,P), that is, a set of players N, a set of winning coalitions W, and a coalition structure P on N.

Given such a game, the corresponding *quotient game* is the simple game (P,W^P) with player set P and set of winning coalitions W^P. A coalition $R \subseteq P$ in the quotient game is winning if and only if the coalition of represented unions $\bigcup_{Q\in R} Q$ is winning in (N,W). We denote the set of minimal winning coalitions in the quotient game by M^P and by M^P_Q the set of minimal winning coalitions containing union $Q \in P$.

In analogy to simple games, we denote simple games with coalition structures by (N,M,P) and the corresponding quotient game by (P,M^P). Thus, a union Q's power in the quotient game, measured by the PGI, amounts to

$$\delta_Q(P,M^P) = \frac{|M^P_Q|}{\sum_k |M^P_{P_k}|}, \quad Q = P_1,\dots,P_p. \tag{2.2}$$

A *coalitional power index* is a mapping f assigning each simple game with coalition structure (N,M,P) an n-dimensional real valued vector $f(N,M,P) = (f_1(N,M,P),\dots, f_n(N,M,P))$.

2.3 *Previous extensions of the PGI for a priori unions*

Alonso-Meijide et al. (2008) introduce two variations of the PGI for a priori unions, the *Solidarity Public Good Index* and the *Owen Extended Public Good Index*. Both indices distribute power in two steps. In the first step, they assign power to each union equal to their PGI in the quotient game (thus satisfying *quotient game property*). In the second step, they use alternative methods to distribute the union's power among its members. The Solidarity Public Good Index Υ does so giving each union member equal power, that is,

$$\Upsilon_i(N,M,P) = \delta_{P(i)}(P,M^P)\frac{1}{|P(i)|}, \quad i = 1,\dots,n. \tag{2.3}$$

Following an idea of Owen (1977), the Owen Extended Public Good Index splits power within unions according to the possibilities that the subsets of this union have to form winning coalitions with other unions. For this purpose, a subset $S \subseteq Q$ of union $Q \in P$ is an *essential part* with respect to coalition $R \in M^P_Q$ if $S \cup \bigcup_{Q'\in R\setminus\{Q\}} Q'$ is a winning coalition in (N,M) and $T \cup \bigcup_{Q'\in R\setminus\{Q\}} Q'$ is losing for all $T \subset S$. Denoting the *set of essential parts* with respect to R containing player i by $E_i^R(N,M,P)$, the Owen Extended Public Good Index Γ is defined as

$$\Gamma_i(N,M,P) = \delta_{P(i)}(P,M^P) \sum_{R \in M^P_{P(i)}} \frac{1}{|M^P_{P(i)}|} \frac{|E_i^R(N,M,P)|}{\sum_{j \in P(i)} |E_j^R(N,M,P)|}, \quad i = 1,\ldots,n. \quad (2.4)$$

3. New variations of the PGI for a priori unions

3.1 *The Union Public Good Index*

The first variation for a priori unions we introduce is the *Union Public Good Index* Λ. As close as possible to the original spirit of the PGI, it is based on the two assumptions that the coalitional value is a public good and only minimal winning coalitions are relevant. The latter assumption does, however, apply to coalitions being minimal also with respect to the coalition structure. A player's power is hence proportional to the number of minimal winning coalitions his union is a member of in the quotient game, that is,

$$\Lambda_i(N,M,P) = \frac{|M^P_{P(i)}|}{\sum_k |P_k||M^P_{P_k}|}, \quad i = 1,\ldots,n. \quad (3.1)$$

As with the Solidarity PGI, it is obviously the case that all members of the same union have equal power, that is, the Union PGI as well satisfies *solidarity*. However, the Union PGI is the only of the overall six extensions not assigning power to unions on the basis of the PGI in the corresponding quotient game.

3.2 *Power distribution based on threats*

There are different approaches to the allocation of power inside unions taking into account the players' threat power in case that they leave their union. Several possibilities arise concerning how the remaining union structure behaves upon withdrawal of a player from a union. While the union might break up into any possible partition, or even the whole union structure might change, we consider the three canonical cases in which (i) the respective union is not more stable than other unions, (ii) all other members of the respective union and (iii) all other unions are treated symmetrically. In either case, at the union level, power is distributed according to the PGI of the quotient game. At the member stage, players receive a share proportional to their power in the the *threat game*, that is, the game induced by their withdrawal from their union.

Assuming the least possible degree of stability, the first variant considers that all unions break up into singletons upon a player's withdrawal from his union. Power inside unions is split by the *Threat PGI* T^1 proportional to the players' power in the game without any unions. This approach turns a blind eye on how the union structure behaves in the long run and, at the same time, takes the reasoning behind the Owen Extended PGI one step further. Now, as it comes to intra-union allocation of power, subsets of a union are not allowed to cooperate with other unions only, but also with subsets of other unions. We thus define

$$T_i^1(N,M,P) = \delta_{P(i)}(P,M^P)\frac{\delta_i(N,M)}{\sum_{j\in P(i)}\delta_j(N,M)}, \quad i=1,\dots,n, \tag{3.2}$$

whenever $\sum_{j\in P(i)}\delta_j(N,M) > 0$ and $T_i^1(N,M,P) = 0$ otherwise.

Presupposing a greater degree of stability, the second variant considers a breakup of the leaving player's union into singletons while all other unions endure. For union $Q \in P$, denote $P/Q = P\backslash\{Q\} \cup \{\{i\}|i \in Q\}$ as the union structure where Q breaks up into singletons $\{i\}$, $i \in Q$, and define the *Threat PGI* T^2 by

$$T_i^2(N,M,P) = \delta_{P(i)}(P,M^P)\frac{\delta_{\{i\}}(P/P(i),M^{P/P(i)})}{\sum_{j\in P(i)}\delta_{\{j\}}(P/P(i),M^{P/P(i)})}, \quad i=1,\dots,n, \tag{3.3}$$

whenever $\sum_{j\in P(i)}\delta_{\{j\}}(P/P(i),M^{P/P(i)}) > 0$ and $T_i^2(N,M,P) = 0$ otherwise.

Finally, the third variant assumes that neither the cooperation of the rest of the union members nor that of other unions is affected by one member's leaving his union. By $P/i = P\backslash\{P(i)\} \cup \{\{i\},P(i)\backslash\{i\}\}$ for a player i and partition P we denote the union structure in which i separates from his union $P(i)$ and plays on his own. Define the *Threat PGI* T^3 by

$$T_i^3(N,M,P) = \delta_{P(i)}(P,M^P)\frac{\delta_{\{i\}}(P/i,M^{P/i})}{\sum_{j\in P(i)}\delta_{\{j\}}(P/j,M^{P/j})}, \quad i=1,\dots,n, \tag{3.4}$$

whenever $\sum_{j\in P(i)}\delta_{\{j\}}(P/j,M^{P/j}) > 0$, and $T_i^3(N,M,P) = \delta_{P(i)}(P,M^P)\frac{1}{|P(i)|}$ otherwise.[1]

[1] Here, a distinction for whether the denominator is 0 or not is necessary. Consider for instance $N = \{1,2,3,4\}$ and $M = \{\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$. With coalition structure

4. Basic properties of coalitional PGIs

Alonso-Meijide et al. (2008) provide axiomatizations of the Solidarity PGI Υ and the Owen Extended PGI Γ. While this does not lie within the scope of our work, we still want to comment on the basic properties and relationships of the coalitional extensions of the Public Good Index discussed here.

To begin with, all PGI extensions are efficient values. While all values but the Union PGI Λ satisfy symmetry among unions, this holds for Λ for equally sized unions only. Symmetry inside unions does, however, always apply. While all indices give a power of zero to null unions, all but the Union PGI Λ and Solidarity PGI Υ necessarily give zero power to null players – the latter two indices satisfy solidarity property because of which null players have positive power if their union is not a null union. In addition, all indices but the Union PGI Λ distribute power in a two-step way such that unions receive overall power as much as assigned by the PGI in the corresponding quotient game (quotient game property).

In order to investigate similarities and differences of the overall six PGI extensions, we start by considering the trivial coalition structures $P^n = \{\{1\},\ldots,\{n\}\}$ and $P^N = \{\{1,\ldots,n\}\}$. In games where the coalition structure is given by singletons (P^n), all coalitional PGIs coincide with the PGI of the game without union structure. However, in the case P^N with one grand union, the Union PGI Λ and the Solidarity PGI Υ amount to the egalitarian power distribution. While the Owen Extended PGI Γ and Threat PGI T^1 coincide with the PGI of the simple game without union structure, no statements can be made concerning the behavior of threat indices T^2 and T^3. In addition, the following can be said about trivial coincidences of the various extensions.

The Union PGI Λ and the Solidarity PGI Υ coincide whenever all unions have equal size.

If at most one union contains more than one member, Threat PGIs T^1 and T^2 coincide.

If at most one union contains two members and all other unions are singletons, threat indices T^1, T^2, and T^3 coincide.

$P = \{\{1\},\{2,3,4\}\}$, it is $M^P = \{\{\{1\},\{2,3,4\}\}\}$ and thus $\delta_{\{2,3,4\}}(P,M^P) = \frac{1}{2}$. However, all members $i \in \{2,3,4\}$ constitute a null union $\{i\}$ with a power of 0 in their respective threat game P/i. Note this distinction was not necessary in the case of indices T^1 and T^2. For both indices it holds that, if all members $i \in Q$ constitute null unions $\{i\}$ in the corresponding threat games, then union Q is a null union itself. Thus, in case the denominator is 0, this also holds for $\delta_Q(P,M^P)$ and hence for the overall expression.

Table 4.1 — Indices for example 4.2.

N	δ	Λ	Υ	Γ	T^1	T^2	T^3
1	$\frac{4}{13}$	$\frac{1}{6}$	$\frac{2}{18}$	$\frac{6}{24}$	$\frac{4}{18}$	$\frac{6}{30}$	$\frac{2}{6}$
2	$\frac{2}{13}$	$\frac{1}{6}$	$\frac{3}{18}$	$\frac{4}{24}$	$\frac{3}{18}$	$\frac{5}{30}$	$\frac{1}{6}$
3	$\frac{2}{13}$	$\frac{1}{6}$	$\frac{3}{18}$	$\frac{4}{24}$	$\frac{3}{18}$	$\frac{5}{30}$	$\frac{1}{6}$
4	$\frac{3}{13}$	$\frac{1}{6}$	$\frac{6}{18}$	$\frac{8}{24}$	$\frac{6}{18}$	$\frac{10}{30}$	$\frac{2}{6}$
5	$\frac{1}{13}$	$\frac{1}{6}$	$\frac{2}{18}$	$\frac{1}{24}$	$\frac{1}{18}$	$\frac{2}{30}$	0
6	$\frac{1}{13}$	$\frac{1}{6}$	$\frac{2}{18}$	$\frac{1}{24}$	$\frac{1}{18}$	$\frac{2}{30}$	0

However, we can find significant differences between the indices as in the following examples.

4.1 Example The possible gap between the Union PGI and the other coalitional PGIs becomes apparent in large, star-formed games (N,M) with set of players $N = \{1,\dots,n\}$ and set of minimal winning coalitions $M = \{\{1,2\},\dots,\{1,n\}\}$. In the corresponding game with coalition structure $P = \{\{1\},\{2,\dots,n\}\}$, the only winning coalition is the grand coalition made up by the two unions $\{1\}$ and $\{2,\dots,n\}$. The Union PGI Λ gives all players in this grand coalition equal power and hence amounts to the egalitarian power distribution (all players having power $\frac{1}{n}$). Satisfying quotient game property as well as symmetry among and inside unions, all other indices assign power $\frac{1}{2}$ to player 1 and $\frac{1}{2(n-1)}$ to the minor players. Note that the overall power of unions thus sums up to $\frac{1}{n}$ for union $\{1\}$ and $\frac{n-1}{n}$ for union $\{2,\dots,n\}$ in the case of Λ, while all other indices assign equal power to the two symmetric unions.

4.2 Example In general, extensions show different aspects of power. Consider for instance the simple game (N,M) given by $N = \{1,\dots,6\}$ and $M = \{\{1,2\},\{1,3\},\{1,4,5\},\{1,4,6\},\{2,3,4\}\}$. A possible representation as a weighted voting game is $(55;35,20,20,15,5,5)$. So (N,M) is not decisive and the PGI $\delta(N,M) = (\frac{4}{13},\frac{2}{13},\frac{2}{13},\frac{3}{13},\frac{1}{13},\frac{1}{13})$ is not monotonic in voting weights. Table 4.1 shows that no two coalitional PGIs are the same for coalition structure $P = \{\{1,5,6\},\{2,3\},\{4\}\}$. Given this coalition structure, any two unions form a minimal winning coalition in the quotient game, that is, $M^P = \{\{\{1,5,6\},\{2,3\}\},\{\{1,5,6\},\{4\}\},\{\{2,3\},\{4\}\}\}$.

Ignoring symmetry between unions but rather accounting for all players being in an equal number of minimal winning coalitions in the quotient game, the Union PGI Λ distributes power in an egalitarian way. The remaining indices respect quotient game and symmetry among unions and hence split power equally between unions. While the Solidarity PGI Υ distributes this power equally among members, the remaining indices do so only for the symmetric members of union $\{2,3\}$. Concerning union $\{1,5,6\}$, the Owen Extended PGI Γ accounts for the essential parts of a player, thus shifting power towards the major player 1. Threat index T^1 does so to a slightly lesser extent, splitting power between members according to their PGI δ. Again somewhat less variability is shown by threat index T^2, allocating power in union $\{1,5,6\}$ proportional to the members' power in the threat game with coalition structure $P/\{1,5,6\} = \{\{1\},\{2,3\},\{4\},\{5\},\{6\}\}$. Finally, the greatest degree of variability is shown by threat index T^3 which allocates power in union $\{1,5,6\}$ proportional to the members' power in the games with coalition structure $P/1$, $P/5$, and $P/6$, respectively. The minor players constitute null unions in the corresponding threat games such that, explaining the minor players' dependence on their union, player 1 receives the whole share of power.

References

Alonso-Meijide, J.M., Casas-Méndez, B., Fiestras-Janeiro, M.G. and Holler, M.J. (2008). Two variations of the Public Good Index for games with a priori unions. mimeo.

Alonso-Meijide, J.M., Casas-Méndez, B., Fiestras-Janeiro, M.G. and Holler, M.J. (2009). The Deegan-Packel index for simple games with a priori unions. *Quality and Quantity*, in print.

Aumann, R. (1977). Game theory. In J. Eatwell, M. M. and Newman, P. (eds.), *The New Palgrave: a Dictionary of Economics*. MacMillan, 460–482.

Holler, M.J. (1982). Forming coalitions and measuring voting power. *Political Studies*, 30: 262–271.

Holler, M.J. and Packel, E. (1983). Power, luck and the right index. *Journal of Economics*, 43: 21–29.

Owen, G. (1977). Values of games with a priori unions. In Henn, R. and Moeschlin, O. (eds.), *Mathematical Economics and Game Theory*. Springer Verlag, 76–88.

Owen, G. (1982). Modification of the Banzhaf-Coleman index for games with a priori unions. In Holler, M.J. (ed.), *Power, Voting and Voting Power*. Physica-Verlag, 232–238.